

**On effective Chern-Simons Term induced
by a Local CPT-Violating Coupling
using γ_5 in Dimensional Regularization ¹**

Ruggero Ferrari^{2ac} and Mario Raciti^{3bc}

^a Center for Theoretical Physics

Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

^bDipartimento di Fisica, Università degli Studi di Milano

^c INFN, Sezione di Milano
via Celoria 16, I-20133 Milano, Italy
(MIT-CTP4680, IFUM-1041-FT, October 2015)

Abstract: We resume a long-standing, yet not forgotten, debate on whether a Chern-Simons birefringence can be generated by a local term $b_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi$ in the Lagrangian (where b_μ are constants).

In the present paper we implement a new way of managing γ_5 in dimensional regularization. Gauge invariance in the underlying theory (QED) is enforced by this choice of defining divergent amplitudes.

We investigate the singular behavior of the vector meson two-point-function around the $m^2 = 0$ and $p^2 = 0$ point. We find that the coefficient of the effective Chern-Simons can be finite or zero. It depends on how one takes the limits: they cannot be interchanged due to the associate change of symmetry.

For $m^2 = 0$ we evaluate also the self-mass of the photon at the second order in b_μ . We find zero.

PACS: 11.10.Gh, 11.30.Rd, 11.40.Ha

¹This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement #DE FG02-05ER41360

²e-mail: ruggferr@mit.edu

³e-mail: mario.raciti@mi.infn.it

1 Introduction

There has been an interesting and long debate (see for instance [1]- [6]) on whether the Lorentz- and CPT-violating Chern-Simons effective action term

$$\Delta\mathcal{S} = \int d^4x \frac{1}{2} c^\mu \varepsilon_{\mu\alpha\beta\gamma} F^{\alpha\beta} A^\gamma \quad (1)$$

might be generated by a local Lorentz- and CPT-violating axial vector term in conventional QED

$$\mathcal{S}_{\text{extended}} = \Lambda^{D-4} \int d^Dx \left[-\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \bar{\psi} \left(i \not{\partial} - e \not{A} - m - \not{b} \gamma_5 \right) \psi \right], \quad (2)$$

where b_μ are constants and Λ is the scale for dimensional regularization.

We discuss this problem at one loop in the perturbative expansion. We use dimensional regularization extended to γ_5 , as briefly presented in the Section 2. The extension γ_5 to generic dimension D has been proposed recently in Refs. [7] and [8]. The choice of dimensional regularization for the present problem is dictated by the need of preserving the gauge invariance of the underlying QED and of profiting of the advantages in actual computations (e.g. the validity of formal properties of integration as: shift in the integration variables and consistent use of divergent integrals).

There have been previous attempts to use dimensional regularization [9]. Our results are in disagreement with the cited paper. The origin of it might depend on the use of different γ_5 extension, i.e. 't Hooft-Veltman's and anticommuting γ_5 . We insist that our approach has many nice properties as cyclicity of the gamma's trace and Lorentz covariance.

Our results are in agreement with previous calculations where QED gauge invariance is enforced through the regularization procedure of Pauli-Villars.

In this framework the one-loop induced Chern-Simons with $m^2 \neq 0$ turns out to vanish in the limit $p^2 = 0$ (on-shell photons). The result removes some ambiguities pointed out in early works and never resolved by the use of dimensional regularization because of the γ_5 problem.

Our detailed analysis shows that the point $m^2 = 0$ $p^2 = 0$ is singular. The limit depends on the sequence in the (m^2, p^2) variables.

We show that this peculiarity is due to a change in the symmetry property of the model. If one puts $m^2 = 0$ right at the beginning the b_μ can be

removed by a local chiral transformation (at the classical level), as noted by many authors. Thus only a local term survives (as in eq. (1)) similar to the ABJ anomaly.

We further investigate the properties of the $m^2 = 0, p^2 \neq 0$ case by evaluating the photon two-point-function at second order of b_μ in the one-loop approximation. We demonstrate that the amplitude must be gauge invariant. The explicit evaluation shows that it is actually null. Thus in this case the quantum corrections do not modify the independence from b_μ of the amplitudes, which is present in the classical approximation.

2 Managing γ_5 in Dimensional Regularization

While the extension of the gamma's to generic D dimensions is considered straightforward since the algebra

$$\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu} \quad (3)$$

doesn't change with D , the situation with γ_5 is much more complex. There is no proof that the extension to non-integer D is possible and, if it is, we do not know how to do it. To mark this important point we use γ_5 for $D = 4$ and γ_χ for its extension (if any). Thus, for instance, we do not know what

$$\{\gamma_\chi, \gamma_\nu\} \quad (4)$$

might be.

The consequences of this fact are important in many respects. One, often neglected, is that the invariance properties of the action change when the gamma algebra is promoted to D dimensions. For instance in $D = 4$ dimensions the kinetic term

$$\int d^4x \bar{\psi} \gamma^\mu \partial_\mu \psi \quad (5)$$

is invariant under global chiral transformations

$$\psi \rightarrow e^{i\alpha\gamma_5} \psi. \quad (6)$$

Therefore its Noether current is conserved (apart possible anomalies). But in generic D dimensions the action

$$\int d^Dx \bar{\psi} \gamma^\mu \partial_\mu \psi \quad (7)$$

might not be invariant under

$$\psi \rightarrow e^{i\alpha\gamma_\chi}\psi \quad (8)$$

because the (anti)commutation relations (4) is not necessarily null.

The second point is the problem of evaluation of the trace, where one or more γ_χ are present. To our experience the two difficulties are deeply intertwined. For instance the anomaly of the axial current cancels the terms responsible of the violation of the invariance of the generating functional. See [7] Sec. 9 for a detailed discussion.

We give a bird view of the approach implemented in the present paper. Some details are not given here, but they can be traced in Refs. [7] and [8].

1. It is assumed that an extension γ_χ exists so that a trace can be defined

$$Tr(p) \equiv Tr\left(\gamma_\chi \not{p}_1 \not{p}_2 \not{p}_3 \dots\right) = Tr\left(\gamma_\chi \gamma^\alpha \gamma^\beta \gamma^\rho \dots\right) p_{1\alpha} p_{2\beta} p_{3\rho} \dots \quad (9)$$

The above expression can be generalized to many γ_χ 's in generic positions and the p 's might be replaced by polarization vectors and by Lorentz covariant tensors, e.g. $g_{\alpha\rho}$ but not $\varepsilon_{\mu\nu\rho\sigma}$. It is important that none of the Lorentz indices are left free: they appear only as dummy variables to sum over without specifying their value (e.g. v_μ and w_μ are not provided, but $v^\mu w_\mu$ is an admissible quantity assuming real or complex values). Lorentz covariance and cyclicity are required.

2. In the neighborhood of $D = 4$ the trace is assumed to admit an expansion

$$Tr(p) = \sum_{h=0} A_h(p) (D-4)^h, \quad h \in \mathcal{N}, \quad (10)$$

where $A_h(p)$ are Lorentz invariants in $D = 4$ dimensions (the tensor $\varepsilon_{\mu\nu\rho\sigma}$ might be present).

3. By assuming that the limit $D = 4$ is smooth, one gets

$$\{\gamma_\chi, \gamma_\mu\} = \mathcal{O}(D-4), \quad \forall \mu. \quad (11)$$

Clearly from step 1. to step 2. the Lorentz symmetry is restricted to $D = 4$.

In the case of a single γ_χ a typical trick, in order to evaluate the trace, is the following (cyclicity is essential)

$$\begin{aligned} & Tr\left(\gamma_\chi \not{p}_1 \not{p}_2 \dots \not{p}_k\right) \\ &= -Tr\left(\not{p}_1 \gamma_\chi \not{p}_2 \dots \not{p}_k\right) + Tr\left(\gamma_\chi \left\{ \not{p}_1, \not{p}_2 \dots \not{p}_k \right\}\right) \end{aligned} \quad (12)$$

The above relation is very intriguing. The first term in the RHS yields the usual gamma's algebra in $D = 4$, while the second provides the $\mathcal{O}(D - 4)$ term.

To summarize, we assume smooth dependence of traces from D around $D = 4$, but at the same time we will avoid the evaluation of any algebra where the (anti)commutation of γ_χ is involved. We might profit of identities like (12). Lorentz covariance and cyclicity are not negotiable. Details are in Refs. [7] and [8], where the ABJ anomaly and the isoscalar anomaly in $SU(2)$ nonabelian chiral gauge theory are explicitly evaluated. In particular in the nonabelian chiral gauge case, the many- γ_χ puzzle is solved.

In the present paper for the calculation we use the template of Ref. [7] with minor and straightforward changes.

3 Peculiarities of the $m^2 = 0$ Case

In the classical approximation the b_μ -term can be removed by a local chiral transformation if $m^2 = 0$.

The argument runs as follows . The generating functional is obtained by path integral on the classical fields

$$Z(J, \eta, \bar{\eta}) = \int \mathcal{D}[A_\mu, \psi, \bar{\psi}] \exp i \int d^4x \left(\bar{\psi} \left(i \not{\partial} - e \not{A} - \not{b} \gamma_5 \right) \psi + J_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta \right), \quad (13)$$

where $J_\mu, \eta, \bar{\eta}$ are external sources. By performing a chiral transformation on the dummy fields (classical and in $D = 4$)

$$\psi \rightarrow e^{-ib_\mu x^\mu \gamma_5} \psi \quad (14)$$

we get

$$\begin{aligned} Z(J, \eta, \bar{\eta}) &= \int \mathcal{D}[A_\mu, \psi, \bar{\psi}] \exp i \int d^4x \left(\bar{\psi} \left(i \not{\partial} - e \not{A} \right) \psi + J_\mu A^\mu + \bar{\eta} e^{-ib_\mu x^\mu \gamma_5} \psi + \bar{\psi} e^{-ib_\mu x^\mu \gamma_5} \eta \right) \\ &= Z_{(0)}(J, e^{-ib_\mu x^\mu \gamma_5} \eta, \bar{\eta} e^{-ib_\mu x^\mu \gamma_5}), \end{aligned} \quad (15)$$

where $Z_{(0)}$ indicates that the action has no b_μ term.

The result in eq. (15) is very strong: if one considers only A_μ -amplitudes there is no dependence from b_μ . Some dependence from b_μ might emerge from quantum corrections as we will see later on.

Before dropping the subject, it is amusing to see how the original action is restored by starting from the identity in (15). For simplicity we drop A_μ . The integration over the Fermi fields yields the usual result

$$Z_{(0)}(\eta, \bar{\eta}) = \exp \left(\int d^4x d^4y \bar{\eta}(x) \frac{i}{(2\pi)^4} \int d^4p \frac{e^{-ip(x-y)}}{\not{p} + i\epsilon} \eta(y) \right). \quad (16)$$

According to eq. (15) we have to replace η by $e^{-ib_\mu x^\mu \gamma_5} \eta$. Use

$$e^{-ib_\mu x^\mu \gamma_5} = e^{-ib_\mu x^\mu} \gamma_+ + e^{ib_\mu x^\mu} \gamma_-, \quad \gamma_\pm \equiv \frac{1 \pm \gamma_5}{2} \quad (17)$$

and get

$$Z_{(0)}(e^{-ib_\mu x^\mu \gamma_5} \eta, \bar{\eta} e^{-ib_\mu x^\mu \gamma_5}) = \exp \left(\int d^4x d^4y \bar{\eta}(x) \frac{i}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \left[\frac{1}{\not{p} - \not{b} + i\epsilon} \gamma_- + \frac{1}{\not{p} + \not{b} + i\epsilon} \gamma_+ \right] \eta(y) \right) \quad (18)$$

With a little algebra the relations can be written in the form

$$\begin{aligned} \frac{1}{\not{p} - \not{b}} \gamma_- &= \left[\left(1 - \not{b} \frac{1}{\not{p}} \right) \not{p} \right]^{-1} \gamma_- = \frac{1}{\not{p}} \left(\sum_{k=0}^{\infty} \left[\not{b} \frac{1}{\not{p}} \right]^k \right) \gamma_- \\ \frac{1}{\not{p} + \not{b}} \gamma_+ &= \left[\left(1 + \not{b} \frac{1}{\not{p}} \right) \not{p} \right]^{-1} \gamma_+ = \frac{1}{\not{p}} \left(\sum_{k=0}^{\infty} \left[-\not{b} \frac{1}{\not{p}} \right]^k \right) \gamma_+ \end{aligned} \quad (19)$$

Finally the sum yields

$$\frac{1}{\not{p} - \not{b}} \gamma_- + \frac{1}{\not{p} + \not{b}} \gamma_+ = \frac{1}{\not{p}} \left(\sum_{k=0}^{\infty} \left[-\not{b} \frac{1}{\not{p}} \gamma_5 \right]^k \right) \quad (20)$$

$$= \frac{1}{\not{p}} \left(1 + \not{b} \frac{1}{\not{p}} \gamma_5 \right)^{-1} = \left[\left(1 + \not{b} \frac{1}{\not{p}} \gamma_5 \right) \not{p} \right]^{-1} = \left(\not{p} - \not{b} \gamma_5 \right)^{-1} \quad (21)$$

i.e. we are back again to the action (13) (by restoring A_μ).

The above example suggests that the correct expansion is in number of loops. At each order in the number of loops the amplitude must be b_μ -independent, when properly summed over all powers of b_μ . This conclusion is unavoidable. Thus the only possibility in order to have a b_μ -dependence of the two-point-function of the vector mesons is by means of an anomaly of the axial current. More precisely: one uses the expansion in powers of b_μ as in equations (20) and (21) for the fermion propagators of the A_μ two-point-function. Among the numerous graphs the triangular ones are expected to be anomalous and therefore a b_μ dependence arises. This will be evaluated in the Section 4.

This is an unexpected situation. The parameter b_μ cannot be fixed phenomenologically by some S-matrix measurement. If the anomaly is measured, we get out the value of b_μ , a fact that is not very much instructive, since the interacting term with fermions disappears (we could use an action with no b_μ).

Comment: In the process of quantization of the field theory (13) one needs a regularization procedure in order to deal with ill-defined amplitudes. We use dimensional regularization and consequently the algebra involving γ_χ is far from trivial (see Section 2). We envisage that classically equivalent Feynman rules yield different results at the quantum level, since there is no possibility to enforce normalization conditions (b_μ disappears from the action at the classical level) to force the theories at the same results. We could use a perturbative expansion where the unperturbed propagator is the conventional free Fermi-Dirac as in eq. (15) or its power series expansion in b_μ (20) or the *non perturbative* approach (21). We choose the power series expansion in b_μ because we do not know how to deal with γ_χ in the denominator in presence of dimensional regularization. Moreover this choice allows a comparison with the existing literature.

4 One-loop Calculations

In this work we give the detailed calculations at one loop of the ensuing terms, due to the b_μ external source given in eq. (2). At the first order we expect a Chern-Simons term, while at the second order a self-mass of the photon might arise.

The result of the computations is quite surprising, thus we present the whole matter systematically. The scenario is the following.

At first order in b_μ and for $p^2 \neq 0, m^2 = 0$ the Chern-Simons term is non zero ($c_\mu \neq 0$ in eq. (1)). The calculation is very close to the derivation of Adler-Bardeen-Jackiw Anomaly [7].

Instead zero is the value of c_μ for $p^2 = 0, m^2 \neq 0$.

For $p^2 \ll m^2 \neq 0$ the coefficient $c_\mu \sim \frac{p^2}{m^2}$. Thus there is a discontinuity at $(p^2 = 0, m^2 = 0)$.

At the second order in b_μ we evaluate the photon self-energy at $m^2 = 0$. We demonstrate that Ward identity is satisfied. The explicit computation yields an indisputable result: zero.

It should be stressed that the use of dimensional regularization is freeing us from problems present in other regularization schemes: there is no ambiguities on shifting by a constant the integration variable of the inner momentum of the loop.

We deal with the γ_χ by following the rules outlined in Section 2. We bring together all factors of the $D - 4$ pole, due to the inner momentum integration. We verify that a common $D - 4$ factor appears to cancel the pole, to yield a final finite result. Only this part of the calculation requires care in the use of γ_χ . The rest of the calculation can be performed at $D = 4$, since every term is finite.

We start from the formulae of Section 8 of the paper [7]. Eventually we take the appropriate limits.

5 ABJ Anomaly ($p^2 \neq 0, m^2 = 0$)

We consider a massive fermion triangle, where one vertex is given by an axial current. We will remove the mass when necessary. Thus we consider the integral (p is the incoming momentum on the vertex σ and k on ρ ; crossed graph will be considered at the end)

$$J_{\mu\rho\sigma}(k, p) = -i\Lambda^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[(q-k)^2 - m^2][q^2 - m^2][(q+p)^2 - m^2]} \\ Tr \left\{ \gamma_\mu \gamma_\chi [(q-k)^\alpha \gamma_\alpha + m] \gamma_\rho [q^\beta \gamma_\beta + m] \gamma_\sigma [(q+p)^\iota \gamma_\iota + m] \right\} \quad (22)$$

Now we use Feynman parameterization and get

$$= -i2\Lambda^{4-D} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \left\{ (q + px - py - ky)^2 - (px - py - ky)^2 \right. \\ \left. - m^2 + p^2(x-y) + k^2 y \right\}^{-3} \\ Tr \left\{ \gamma_\mu \gamma_\chi [(q-k)^\alpha \gamma_\alpha + m] \gamma_\rho [q^\beta \gamma_\beta + m] \gamma_\sigma [(q+p)^\iota \gamma_\iota + m] \right\}. \quad (23)$$

We change variable

$$q \rightarrow q + r, \quad r \equiv yk - xp + yp \quad (24)$$

and implement symmetric integration

$$= -i2\Lambda^{4-D} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \left\{ q^2 - (px - py - ky)^2 \right.$$

$$\begin{aligned}
& -m^2 + p^2(x-y) + k^2y \Big\}^{-3} \\
& Tr \left\{ \gamma_\mu \gamma_\chi [(q+r-k)^\alpha \gamma_\alpha + m] \gamma_\rho [(q+r)^\beta \gamma_\beta + m] \gamma_\sigma \right. \\
& \left. [(q+r+p)^\iota \gamma_\iota + m] \right\}.
\end{aligned} \tag{25}$$

Let us define

$$\Delta = m^2 - p^2(x-y) - k^2y + (px - py - ky)^2 \tag{26}$$

Thus

$$\begin{aligned}
& = -i2\Lambda^{4-D} \int_0^1 dx \int_0^x dy \int \frac{d^D q}{(2\pi)^D} \left\{ q^2 - \Delta \right\}^{-3} \\
& Tr \left\{ \gamma_\mu \gamma_\chi [(q+yk-xp+yp-k)^\alpha \gamma_\alpha + m] \right. \\
& \gamma_\rho [(q+yk-xp+yp)^\beta \gamma_\beta + m] \gamma_\sigma \\
& \left. [(q+yk-xp+yp+p)^\iota \gamma_\iota + m] \right\}.
\end{aligned} \tag{27}$$

After symmetric integration over q we can split the integral into a divergent part (the terms proportional to m are associated to a Tr with an odd number of gamma's and one γ_χ ; this is expected to be zero for $D \sim 4$)

$$\begin{aligned}
J_{\mu\rho\sigma}^{\text{DIV}}(k, p) & = -i\frac{1}{2} \int_0^1 dx \int_0^x dy Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\
& \left((yk-xp+yp+p)^\iota g^{\alpha\beta} + (yk-xp+yp)^\beta g^{\alpha\iota} + (yk-xp+yp-k)^\alpha g^{\beta\iota} \right) \\
& \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(\Delta) \right] \left(1 - \frac{D-4}{4} - (D-4) \ln \Lambda \right)
\end{aligned} \tag{28}$$

and finite part

$$\begin{aligned}
J_{\mu\rho\sigma}^{\text{CONV}}(k, p) & = -i2 \int_0^1 dx \int_0^x dy \frac{-i}{2(4\pi)^2} \frac{1}{\Delta} \\
& Tr \left\{ \gamma_\mu \gamma_\chi [(yk-xp+yp-k)^\alpha \gamma_\alpha + m] \gamma_\rho [(yk-xp+yp)^\beta \gamma_\beta + m] \gamma_\sigma \right. \\
& \left. [(yk-xp+yp+p)^\iota \gamma_\iota + m] \right\}.
\end{aligned} \tag{29}$$

In front of the two amplitudes (28) and (29) the gamma's trace must be expanded in powers of $(D-4)$ as required by eq. (22). For the finite part in eq. (29) we can use the $D=4$ expression, but for the divergent part one needs also the linear part in $(D-4)$.

For $D \rightarrow 4$ we get

$$\begin{aligned} & \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - m^2)^3} \\ &= -\frac{i}{(4\pi)^2} \left[\frac{2}{D-4} + \gamma + \frac{1}{2} + \ln\left(\frac{m^2}{4\pi}\right) \right] + \mathcal{O}(D-4) \end{aligned} \quad (30)$$

and

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2)^3} = -\frac{i}{2(4\pi)^2} \frac{1}{m^2} + \mathcal{O}(D-4). \quad (31)$$

The splitting in divergent and convergent parts (28) and (29) is arbitrary. It is done only in order to make the exposition simpler.

5.1 Calculation of Divergent Part ($p^2 \neq 0, m^2 = 0$)

Now we consider to the limit $m^2 = 0$. Since we look at the effective Chern-Simons term we take the conditions

$$k = -p. \quad (32)$$

The divergent part (28) becomes

$$\begin{aligned} J_{\mu\rho\sigma}^{\text{DIV}}(p) &= -i\frac{1}{2} \int_0^1 dx \int_0^x dy \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\ & \left((-xp + p)^\iota g^{\alpha\beta} + (-xp)^\beta g^{\alpha\iota} + (-xp + p)^\alpha g^{\beta\iota} \right) \\ & \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} - \frac{1}{2} - 2 \ln \Lambda \right. \\ & \left. + \gamma + 2 - \ln 4\pi + \ln(-p^2(x - x^2)) \right] \end{aligned} \quad (33)$$

The trace is evaluated according to the rules of Ref. [7] Sec.8.

$$\begin{aligned} & -i\frac{1}{2} \int_0^1 dx \int_0^x dy \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\ & \left((1-x)(2-D) + x(6-D) + (1-x)(2-D) \right) \\ & \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} - \frac{1}{2} - 2 \ln \Lambda \right. \\ & \left. + \gamma + 2 - \ln 4\pi + \ln(-p^2(x - x^2)) \right] \\ &= -i\frac{1}{2} \int_0^1 dx x \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\ & \left(6x - 4 + (D-4)(x-2) \right) \end{aligned}$$

$$\left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} - \frac{1}{2} - 2 \ln \Lambda \right. \\ \left. + \gamma + 2 - \ln 4\pi + \ln(-p^2(x-x^2)) \right] \quad (34)$$

Since $\int_0^1 dx x(6x-4) = 0$ the x -independent part inside the square brackets gives zero contribution. What is left is

$$= -i \left(-\frac{i}{(4\pi)^2} \right) \frac{1}{2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\ \left(\int_0^1 dx (6x^2 - 4x) \ln(x-x^2) - \frac{4}{3} \right) \quad (35)$$

Elementary integration yields

$$\int_0^1 dx (6x^2 - 4x) \ln x(1-x) = -\frac{1}{3} \quad (36)$$

Finally eq. (35) becomes

$$J_{\mu\rho\sigma}^{\text{DIV}}(p) = \frac{1}{(4\pi)^2} \frac{1}{2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \frac{5}{3} \quad (37)$$

Notice that the scale parameter Λ has dropped out from the final result.

5.2 Calculation of convergent Part ($p^2 \neq 0, m^2 = 0$)

Now we evaluate the convergent part (29) in the particular kinematic condition (32). Every integral is finite; therefore the algebra is in $D = 4$

$$J_{\mu\rho\sigma}^{\text{CONV}}(p) = -i2 \int_0^1 dx x \frac{-i}{2(4\pi)^2} \frac{1}{-p^2(x-x^2)} \\ \text{Tr} \left\{ \gamma_\mu \gamma_\chi [(-xp+p)^\alpha \gamma_\alpha] \gamma_\rho [(-xp)^\beta \gamma_\beta] \gamma_\sigma \right. \\ \left. [(-xp+p)^\iota \gamma_\iota] \right\} \\ = \int_0^1 dx x \frac{1}{(4\pi)^2} \frac{1}{p^2(x-x^2)} \\ \text{Tr} \left\{ \gamma_\mu \gamma_\chi p^\alpha \gamma_\alpha (-x+1) \gamma_\rho (-x) p^\beta \gamma_\beta \gamma_\sigma \right. \\ \left. (-x+1) p^\iota \gamma_\iota \right\} \\ = \int_0^1 dx x \frac{1}{(4\pi)^2} \frac{1}{p^2(x-x^2)}$$

$$\begin{aligned}
& Tr \left\{ \gamma_\mu \gamma_\chi p^2 (-x+1) \gamma_\rho x \gamma_\sigma (-x+1) p^\ell \gamma_\iota \right\} \\
&= \int_0^1 dx x \frac{1}{(4\pi)^2} (1-x) Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\ell \gamma_\iota \right\} \\
&= \frac{1}{(4\pi)^2} \frac{1}{6} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\ell.
\end{aligned} \tag{38}$$

5.3 Final Result for $m^2 = 0$

We add the divergent and finite parts in eqs.(37) and (38) and their crossed terms

$$\begin{aligned}
& J_{\mu\rho\sigma}^{\text{DIV}}(p) + J_{\mu\rho\sigma}^{\text{CONV}}(p) + J_{\mu\sigma\rho}^{\text{DIV}}(-p) + J_{\mu\sigma\rho}^{\text{CONV}}(-p) \\
&= 2 \frac{1}{(4\pi)^2} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\ell.
\end{aligned} \tag{39}$$

This is consistent with the ABJ anomaly of the axial current, i.e. one gets the current from the $k \neq -p$ anomaly and then takes the limit value $k = -p$.

Thus we conclude that the CPT-violating term in the action (2) induces a Cern-Simons effective amplitude (1) with

$$c_\mu = \frac{b_\mu}{4\pi^2}, \tag{40}$$

by taking $Tr\{\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\} = i4\varepsilon_{\mu\nu\rho\sigma}$.

The result in eq. (40) has been discussed at length in the literature. See for instance Refs. [10] and [11].

6 Induced Chern-Simons for $p^2 = 0, m^2 \neq 0$

For comparison with the Section 5 and for phenomenological applications it is convenient now to discuss the case $p^2 = 0, m^2 \neq 0$.

We consider the divergent part (28) at $p^2 = 0$

$$\begin{aligned}
T_{\mu\rho\sigma}^{\text{DIV}} &= -i \frac{2}{D} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \left(\frac{1}{6} g^{\alpha\beta} p^\ell - \frac{1}{3} g^{\alpha\iota} p^\beta + \frac{1}{6} g^{\beta\iota} p^\alpha \right) \\
&\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(m^2) \right] \\
&= -i \frac{2}{D} \left(\frac{2-D}{6} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\ell \right. \\
&\quad \left. - \frac{6-D}{3} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\beta \gamma_\sigma \right\} p^\beta + \frac{2-D}{6} Tr \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\sigma \right\} p^\alpha \right) \\
&\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(m^2) \right]
\end{aligned}$$

$$\begin{aligned}
&= -i \frac{2}{D} \frac{2}{3} (4-D) \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\
&\left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(m^2) \right] \\
&= \frac{2}{3} \frac{1}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota
\end{aligned} \tag{41}$$

6.1 Calculation of Convergent Part $p^2 = 0, m^2 \neq 0$.

Thus eq. (29) becomes (at $p^2 = 0$)

$$\begin{aligned}
T_{\mu\rho\sigma}^{\text{CONV}} &= -i2 \int_0^1 dx (-1) \frac{i}{2(4\pi)^2} \frac{1}{m^2} \\
&m^2 \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma (1-x) p^\iota \gamma_\iota + \gamma_\mu \gamma_\chi \gamma_\rho (-1) x p^\beta \gamma_\beta \gamma_\sigma \right. \\
&\left. + \gamma_\mu \gamma_\chi (1-x) p^\alpha \gamma_\alpha \gamma_\rho \gamma_\sigma \right\} \\
&= -\frac{1}{(4\pi)^2} \int_0^1 dx \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota (1-x + x + 1-x) \\
&= -\frac{1}{(4\pi)^2} \int_0^1 dx (2x - x^2) \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\
&= -\frac{1}{(4\pi)^2} \left(1 - \frac{1}{3}\right) \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \\
&= -\frac{2}{3} \frac{1}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota.
\end{aligned} \tag{42}$$

The crossed graph gives a factor 2

$$T_{\mu\rho\sigma}^{\text{CONV+CROSSED}} = -\frac{4}{3} \frac{1}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota. \tag{43}$$

We add the divergent part eq. (41) (notice the difference with Section 8.1 of [7] with $k = -p$) and the crossed divergent part

$$T_{\mu\rho\sigma}^{\text{DIV+CROSSED}} = \frac{1}{(4\pi)^2} \frac{4}{3} \text{Tr} \left(\gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right) p^\iota. \tag{44}$$

Finally one gets

$$T_{\mu\rho\sigma}^{\text{CONV+CROSSED}} + T_{\mu\rho\sigma}^{\text{DIV+CROSSED}} = 0. \tag{45}$$

The result is surprisingly different from the one obtained in Section 5 in eq. (39). The difference between $m^2 = 0$ and $m^2 \neq 0$ has been discussed also in Ref. [12].

Thus we investigate the dependence (c_μ) from p^2 in a massive theory.

7 Photon Two-point Function at $|p^2| \ll m^2$

For virtual photons it is relevant to evaluate the two-point function off-shell. However we hold the condition

$$|p^2| \ll m^2. \quad (46)$$

The general case is discussed in Appendix C.

For the present case we resume eqs. (28) and (29) and expand in $\frac{p^2}{m^2}$. The divergent part yields

$$\begin{aligned} P_{\mu\rho\sigma}^{\text{DIV}} &= -i \frac{2}{D} \int_0^1 x dx \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\ &\quad \left((1-x) g^{\alpha\beta} p^\iota - x g^{\alpha\iota} p^\beta + (1-x) g^{\beta\iota} p^\alpha \right) \\ &\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(m^2) - \frac{p^2}{m^2} (x-x^2) \right] \end{aligned} \quad (47)$$

and finite part

$$\begin{aligned} P_{\mu\rho\sigma}^{\text{CONV}} &= -i 2 \int_0^1 x dx \frac{-i}{2(4\pi)^2} \frac{1}{m^2 - p^2(x-x^2)} \\ &\quad \text{Tr} \left\{ \gamma_\mu \gamma_\chi [(1-x)p^\alpha \gamma_\alpha + m] \gamma_\rho [-xp^\beta \gamma_\beta + m] \gamma_\sigma \right. \\ &\quad \left. [(1-x)p^\iota \gamma_\iota + m] \right\} \\ &= -\frac{1}{(4\pi)^2} \int_0^1 x dx \left(\frac{1}{m^2} + (x-x^2) \frac{p^2}{m^4} \right) \\ &\quad \text{Tr} \left\{ \gamma_\mu \gamma_\chi [(1-x)p^\alpha \gamma_\alpha + m] \gamma_\rho [-xp^\beta \gamma_\beta + m] \gamma_\sigma \right. \\ &\quad \left. [(1-x)p^\iota \gamma_\iota + m] \right\}. \end{aligned} \quad (48)$$

7.1 Photon Two-point Function at $|p^2| \ll m^2$: Divergent Part

We elaborate eq. (47). We keep only the p^2 dependence

$$\begin{aligned} P_{\mu\rho\sigma}^{\text{DIV}} &= \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\ &\quad \int_0^1 x dx \left((1-x) g^{\alpha\beta} p^\iota - x g^{\alpha\iota} p^\beta + (1-x) g^{\beta\iota} p^\alpha \right) (x-x^2) \\ &= \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \end{aligned}$$

$$\begin{aligned}
& \int_0^1 dx \left((x^2 - 2x^3 + x^4)g^{\alpha\beta}p^\iota - (x^3 - x^4)g^{\alpha\iota}p^\beta + (x^2 - 2x^3 + x^4)g^{\beta\iota}p^\alpha \right) \\
&= \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\
& \left(\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) g^{\alpha\beta} p^\iota - \frac{1}{20} g^{\alpha\iota} p^\beta + \frac{1}{30} g^{\beta\iota} p^\alpha \right) \\
&= \frac{1}{10} \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\
& \left(\frac{1}{3} g^{\alpha\beta} p^\iota - \frac{1}{2} g^{\alpha\iota} p^\beta + \frac{1}{3} g^{\beta\iota} p^\alpha \right) \\
&= \frac{1}{10} \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \\
& \left(\frac{2}{3} (2-D) \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota - \frac{1}{2} (6-D) \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\beta \gamma_\sigma \right\} p^\beta \right) \\
&= \frac{1}{10} \frac{1}{(4\pi)^2} \frac{2}{D} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \left(-\frac{4}{3} + 1 \right) \\
&= -\frac{1}{60} \frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\iota \right\} p^\iota \tag{49}
\end{aligned}$$

7.2 Photon Two-point Function at $|p^2| \ll m^2$: Convergent Part

In the finite part (48) we have two contributions: one from p^3 and one from m^2 .

$$\begin{aligned}
P_{\mu\rho\sigma}^{\text{CONV}} &= - \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{1}{m^2} + (x-x^2) \frac{p^2}{m^4} \right) \\
& \text{Tr} \left\{ \gamma_\mu \gamma_\chi [(1-x)p^\alpha \gamma_\alpha + m] \gamma_\rho [-xp^\beta \gamma_\beta + m] \gamma_\sigma \right. \\
& \left. [(1-x)p^\iota \gamma_\iota + m] \right\} \\
&= -\frac{1}{(4\pi)^2} \int_0^1 dx \left(-x(1-x)^2 \frac{1}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi p^\alpha \gamma_\alpha \gamma_\rho p^\beta \gamma_\beta \gamma_\sigma p^\iota \gamma_\iota \right\} \right. \\
& \left. + (x-x^2) \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \left[(1-x) \gamma_\rho \gamma_\sigma p^\iota \gamma_\iota - x \gamma_\rho p^\beta \gamma_\beta \gamma_\sigma + (1-x) p^\alpha \gamma_\alpha \gamma_\rho \gamma_\sigma \right] \right\} \right) \\
&= -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\iota \gamma_\iota \right\} \int_0^1 dx \left[x^2(1-x)^2 + x^2(1-x) \left((1-x) + x + (1-x) \right) \right] \\
&= -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\iota \gamma_\iota \right\} \int_0^1 dx (x^2 - 2x^3 + x^4 + 2x^2 - 3x^3 + x^4) \\
&= -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\iota \gamma_\iota \right\} \left(1 - \frac{5}{4} + \frac{2}{5} \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\nu \gamma_\nu \right\} \frac{20 - 25 + 8}{20} \\
&= -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\nu \gamma_\nu \right\} \frac{3}{20}.
\end{aligned} \tag{50}$$

7.3 Total

$$P_{\mu\rho\sigma}^{\text{DIV}} + P_{\mu\rho\sigma}^{\text{CONV}} = -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\nu \gamma_\nu \right\} \frac{10}{60} \tag{51}$$

Now we add the crossed term (multiply by 2)

$$P_{\mu\rho\sigma}^{\text{DIV+CROSSED}} + P_{\mu\rho\sigma}^{\text{CONV+CROSSED}} = -\frac{1}{(4\pi)^2} \frac{p^2}{m^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma p^\nu \gamma_\nu \right\} \frac{1}{3}. \tag{52}$$

The result in eq. (52) shows that there is continuity for $p^2 \rightarrow 0$ with the value (45) of Section 6.

8 Second Order in b_μ : Self-Energy with $m^2 = 0$.

The calculation in Section 5 shows that a finite result emerge from the conspiring of pole and zero in $D - 4$; although the b_μ term in the action can be removed by a chiral transformation as in eq. (15).

It is natural to look for other cases where the Feynman integrals provide poles in $(D - 4)$. Thus we consider the photon two-point-function at the second order in b_μ . Here we have three graphs. One $(\Pi_{\rho\sigma}^{(1)})$ where \not{b} alternates with \not{A} on the fermion line and $(\Pi_{\rho\sigma}^{(2)})$ and $(\Pi_{\rho\sigma}^{(3)})$ where \not{b} are consecutive.

Transversality is important issue since the amplitude is a photon self-energy.

8.1 Gauge Invariance at 1-loop: the Box ($m^2 = 0$).

This important question must be discussed in detail.

One expects no problems since the Ward identity should not be affected by the CPT -violating term. I.e. the generating functional should obey the transversality condition. The presence of γ_χ is irrelevant for the derivation of the functional identity.

It is however intriguing to see it in an elementary derivation, i.e. by using

$$p^\rho \frac{1}{[\not{q} + i\varepsilon]} \gamma_\rho \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} = \frac{1}{[\not{q} + i\varepsilon]} - \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} \tag{53}$$

Let us consider the box diagrams one by one.

$$\prod_{\rho\sigma}(p)^{(1)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \frac{1}{[\not{q} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{\not{q} + i\varepsilon} \gamma_\rho \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} \right. \\ \left. \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \gamma_\sigma \right\} \quad (54)$$

Then to check gauge invariance we evaluate

$$p^\rho \prod_{\rho\sigma}(p)^{(1)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \frac{1}{[\not{q} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{\not{q} + i\varepsilon} \right. \\ \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \gamma_\sigma \\ \left. - \frac{1}{[\not{q} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \gamma_\sigma \right\} \quad (55)$$

Now we consider the graphs where both b_μ insertions are on the same Fermionic line

$$\prod_{\rho\sigma}(p)^{(2)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \gamma_\sigma \frac{1}{\not{q} + i\varepsilon} \gamma_\rho \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \not{b}\gamma_\chi \right. \\ \left. \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \right\} \quad (56)$$

Then the divergence gives

$$p^\rho \prod_{\rho\sigma}(p)^{(2)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \gamma_\sigma \frac{1}{\not{q} + i\varepsilon} \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \not{b}\gamma_\chi \right. \\ \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \\ \left. - \gamma_\sigma \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \right\}. \quad (57)$$

Finally the third graph (obtained from $\prod_{\rho\sigma}(p)^{(2)}$ by $\rho \leftrightarrow \sigma, p \rightarrow -p$) gives

$$\prod_{\rho\sigma}(p)^{(3)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \gamma_\rho \frac{1}{\not{q} + i\varepsilon} \gamma_\sigma \frac{1}{[(\not{q} - \not{p}) + i\varepsilon]} \not{b}\gamma_\chi \right. \\ \left. \frac{1}{[\not{q} - \not{p} + i\varepsilon]} \not{b}\gamma_\chi \frac{1}{[\not{q} - \not{p} + i\varepsilon]} \right\} \quad (58)$$

and its divergence

$$p^\rho \prod_{\rho\sigma}(p)^{(3)} = - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \gamma_\sigma \frac{1}{[(\not{q} - \not{p}) + i\varepsilon]} \not{b}\gamma_\chi \right.$$

$$\left\{ \frac{1}{[\not{q} - \not{p} + i\varepsilon]} \not{p} \gamma_\chi \frac{1}{[\not{q} - \not{p} + i\varepsilon]} - \frac{1}{\not{q} + i\varepsilon} \gamma_\sigma \frac{1}{[(\not{q} - \not{p}) + i\varepsilon]} \not{p} \gamma_\chi \frac{1}{[\not{q} - \not{p} + i\varepsilon]} \not{p} \gamma_\chi \right\} \quad (59)$$

The equation above (55), (57) and (59) show the exact cancellation of the divergence of the two-point-function

$$p^\rho \left(\prod_{\rho\sigma} (p)^{(1)} + \prod_{\rho\sigma} (p)^{(2)} + \prod_{\rho\sigma} (p)^{(3)} \right) = 0 \quad (60)$$

In particular the second in eq. (57) cancels the first in eq. (59) through a change of variable. The first in (55) cancels the second in (59). Finally the second in (55) cancels the first in (57). No γ_χ was involved in the proof.

However there is a flaw in the proof: the quantities present in the eqs. (55), (57) and (59) are ill-defined due to ultraviolet divergences: the algebra of γ_χ is needed for generic D . The verifications of the equations will be possible after a precise statement about the algebra.

Only when we group all the $D-4$ poles together we obtain a well-defined quantity. At that moment we can verify the Ward identity.

Transversality then requires the following general form

$$\begin{aligned} \prod_{\rho\sigma} (p) &= a_1 b^2 \left[p_\rho p_\sigma - p^2 g_{\rho\sigma} \right] \\ &+ \frac{a_2}{p^2} \left[p^2 b_\rho b_\sigma + (pb)^2 g_{\rho\sigma} - (pb)(p_\rho b_\sigma + b_\rho p_\sigma) \right]. \end{aligned} \quad (61)$$

8.2 Divergent Parts: the Box $m^2 = 0$.

We look at the divergent parts (by this we denote the amplitude where a q^4 power appears in the numerator). Thus we do not consider the difference between eq. (98) and (99). Later on we will take into account also the p^2 dependence.

From eq. (54) we have the gamma's factor

$$\begin{aligned} \mathcal{V}_{\rho\sigma}^{(1)\text{DIV}} &= \text{Tr} \left\{ \gamma_\rho \not{q} \not{p} \gamma_\chi \not{q} \gamma_\sigma \not{q} \not{p} \gamma_\chi \not{q} \right\} \\ &= \text{Tr} \left\{ (-q^2 \gamma_\rho + 2q_\rho \not{q}) \not{p} \gamma_\chi (-q^2 \gamma_\sigma + 2q_\sigma \not{q}) \not{p} \gamma_\chi \right\} \\ &= q^4 \text{Tr} \left\{ \gamma_\rho \not{p} \gamma_\chi \gamma_\sigma \not{p} \gamma_\chi \right\} - 2q_\rho q^2 \text{Tr} \left\{ \not{q} \not{p} \gamma_\chi \gamma_\sigma \not{p} \gamma_\chi \right\} \\ &\quad - 2q_\sigma q^2 \text{Tr} \left\{ \gamma_\rho \not{p} \gamma_\chi \not{q} \not{p} \gamma_\chi \right\} + 4q_\rho q_\sigma \text{Tr} \left\{ \not{q} \not{p} \gamma_\chi \not{q} \not{p} \gamma_\chi \right\} \end{aligned} \quad (62)$$

The symmetric integration gives

$$\begin{aligned}
\mathcal{V}_{\rho\sigma}^{(1)\text{DIV}} &= q^4 \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} \\
&- \frac{2}{D} q^4 \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} - \frac{2}{D} q^4 \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} \\
&+ \frac{4q^4}{D(D+2)} \left(g_{\rho\sigma} \text{Tr} \left\{ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \right\} + 2 \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} \right) \\
&= q^4 \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} \left(1 - \frac{4}{D} + \frac{8}{D(D+2)} \right) \\
&+ \frac{4q^4}{D(D+2)} g_{\rho\sigma} \text{Tr} \left\{ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \right\} \\
&= \frac{(D-2)q^4}{D+2} \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\sigma \not{\epsilon}_{\gamma_X} \right\} + \frac{4q^4 g_{\rho\sigma}}{D(D+2)} \text{Tr} \left\{ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \right\} \quad (63)
\end{aligned}$$

The other graphs yield

$$\begin{aligned}
\mathcal{V}_{\rho\sigma}^{(2)\text{DIV}} &= \text{Tr} \left\{ \not{\epsilon}_{\gamma_\rho} \not{\epsilon}_{\gamma_X} \not{\epsilon}_{\gamma_X} \not{\epsilon}_{\gamma_\sigma} \right\} \\
&= \text{Tr} \left\{ \left[2q_\rho \not{\epsilon} - q^2 \gamma_\rho \right] \not{\epsilon}_{\gamma_X} \not{\epsilon}_{\gamma_X} \not{\epsilon}_{\gamma_\sigma} \right\} \quad (64)
\end{aligned}$$

By symmetric integration

$$\begin{aligned}
\mathcal{V}_{\rho\sigma}^{(2)\text{DIV}} &= \frac{2q^4}{D(D+2)} \text{Tr} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \gamma_\sigma + (2-D) \gamma_\sigma \not{\epsilon}_{\gamma_X} \gamma_\rho \not{\epsilon}_{\gamma_X} \right. \\
&+ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \gamma_\rho \gamma_\sigma \left. \right\} - \frac{q^4}{D} \left\{ \gamma_\rho \not{\epsilon}_{\gamma_X} \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \gamma_\sigma \right\} \\
&= -\frac{q^4}{D+2} \text{Tr} \left\{ \not{\epsilon}_{\gamma_X} \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \gamma_\sigma \gamma_\rho \right\} + \frac{2q^4(2-D)}{D(D+2)} \text{Tr} \left\{ \gamma_\sigma \not{\epsilon}_{\gamma_X} \gamma_\rho \not{\epsilon}_{\gamma_X} \right\} \\
&+ \frac{2q^4}{D(D+2)} \text{Tr} \left\{ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \gamma_\rho \gamma_\sigma \right\} \quad (65)
\end{aligned}$$

We add the third graph ($\rho \leftrightarrow \sigma$)

$$\begin{aligned}
&\mathcal{V}_{\rho\sigma}^{(2)\text{DIV}} + \mathcal{V}_{\rho\sigma}^{(3)\text{DIV}} \\
&= -\frac{2q^4}{D+2} g_{\rho\sigma} \text{Tr} \left\{ \not{\epsilon}_{\gamma_X} \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \right\} \\
&+ \frac{4q^4(2-D)}{D(D+2)} \text{Tr} \left\{ \gamma_\sigma \not{\epsilon}_{\gamma_X} \gamma_\rho \not{\epsilon}_{\gamma_X} \right\} \\
&+ \frac{4q^4}{D(D+2)} g_{\rho\sigma} \text{Tr} \left\{ \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau \not{\epsilon}_{\gamma_X} \right\} \\
&= \frac{2q^4(2-D)}{D(D+2)} \text{Tr} \left\{ g_{\rho\sigma} \not{\epsilon}_{\gamma_X} \gamma_\tau \not{\epsilon}_{\gamma_X} \gamma^\tau + 2 \gamma_\sigma \not{\epsilon}_{\gamma_X} \gamma_\rho \not{\epsilon}_{\gamma_X} \right\} \quad (66)
\end{aligned}$$

8.3 Pole Part: the Box $m^2 = 0$.

Now we take the sum

$$\begin{aligned}
\mathcal{V}_{\rho\sigma}^{\text{DIV}} &= \mathcal{V}_{\rho\sigma}^{(1)\text{DIV}} + \mathcal{V}_{\rho\sigma}^{(2)\text{DIV}} + \mathcal{V}_{\rho\sigma}^{(3)\text{DIV}} \\
&= -2q^4 g_{\rho\sigma} \frac{(D-4)}{D(D+2)} \text{Tr} \left\{ \gamma_\tau \not{b}_{\gamma_\chi} \gamma^\tau \not{b}_{\gamma_\chi} \right\} \\
&\quad + q^4 \frac{(D-4)(D-2)}{D(D+2)} \text{Tr} \left\{ \gamma_\sigma \not{b}_{\gamma_\chi} \gamma_\rho \not{b}_{\gamma_\chi} \right\}
\end{aligned} \tag{67}$$

It vanishes for $D = 4$; then we can use the elementary algebra ⁴

$$\begin{aligned}
\mathcal{V}_{\rho\sigma}^{\text{DIV}} &= 2q^4 g_{\rho\sigma} b^2 \frac{(D-4)(D-2)}{D(D+2)} \\
&\quad + q^4 \frac{(D-2)}{D(D+2)} (D-4) (2b_\rho b_\sigma - b^2 g_{\rho\sigma}) \\
&= q^4 \frac{(D-4)(D-2)}{D(D+2)} (2b_\rho b_\sigma + b^2 g_{\rho\sigma})
\end{aligned} \tag{68}$$

Finally we get the pole part in $D - 4$, from eqs. (54), (56), (58), (68), (98), (99) and (108)

$$\prod_{\rho\sigma}^{\text{POLE}} = \frac{i}{(4\pi)^2} \frac{1}{6} (2b_\rho b_\sigma + b^2 g_{\rho\sigma}) \tag{69}$$

8.4 Log Parts: the Box $m^2 = 0$.

For the log parts we need the integrals in (98) and (99) The graph n. 1 yields the following divergent-log contribution. From eq. (98), (63) (108) and (109) we have

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{LOG}(1)} &= \frac{1}{3} \left(\text{Tr} \left\{ \gamma_\rho \not{b}_{\gamma_\sigma} \not{b} \right\} - g_{\rho\sigma} \text{Tr} \left\{ \not{b} \not{b} \right\} \right) \\
&\quad - 6 \int_0^1 dx \int_0^x dy y \left(-\frac{i}{(4\pi)^2} \right) \ln(y(1-y)) \\
&= \frac{2}{3} (b_\rho b_\sigma - b^2 g_{\rho\sigma}) \left(-\frac{5}{3} \right) \left(\frac{i}{(4\pi)^2} \right) \\
&= -\frac{i}{(4\pi)^2} \frac{10}{9} (b_\rho b_\sigma - b^2 g_{\rho\sigma})
\end{aligned} \tag{70}$$

⁴A factor $\text{Tr}\{\mathcal{I}\}$ will be neglected throughout the paper.

For the graph n.2 we have similarly the following contributions. Eqs. (99), (65), (108) and (112) are needed

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{LOG}^{(2)}} &= -\frac{1}{6} \left(\text{Tr} \left\{ \gamma_\rho \not{b} \gamma_\sigma \not{b} \right\} - g_{\rho\sigma} \text{Tr} \left\{ \not{b} \not{b} \right\} \right) \\
&- 6 \int_0^1 dx \frac{x^2}{2} \left(-\frac{i}{(4\pi)^2} \right) \ln(x(1-x)) \\
&= -\frac{1}{3} (b_\rho b_\sigma - b^2 g_{\rho\sigma}) \left(-\frac{13}{6} \right) \left(\frac{i}{(4\pi)^2} \right) \\
&= \frac{i}{(4\pi)^2} \frac{13}{18} (b_\rho b_\sigma - b^2 g_{\rho\sigma})
\end{aligned} \tag{71}$$

Similar result for graph 3

$$\prod_{\rho\sigma}^{\text{LOG}^{(3)}} = \frac{i}{(4\pi)^2} \frac{13}{18} (b_\rho b_\sigma - b^2 g_{\rho\sigma}) \tag{72}$$

Then the total divergent-log is

$$\sum_{j=1}^3 \prod_{\rho\sigma}^{\text{LOG}^{(j)}} = \frac{i}{(4\pi)^2} \frac{1}{3} (b_\rho b_\sigma - b^2 g_{\rho\sigma}) \tag{73}$$

The total divergent part is then (eqs. (69) and (73))

$$\prod_{\rho\sigma}^{\text{DIV}} = \frac{i}{(4\pi)^2} \frac{1}{6} (4b_\rho b_\sigma - b^2 g_{\rho\sigma}) \tag{74}$$

8.5 Convergent Parts: Graph 1

After the shift suggested by eq. (98) eq. (54) becomes (γ_χ dropped)

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}^{(1)}} (p) &= -6 \int_0^1 dx \int_0^x dy y \int \frac{d^D q}{(2\pi)^D} \left[q^2 + p^2 y(1-y) \right]^{-4} \\
&\text{Tr} \left\{ \gamma_\rho (\not{q} - y \not{p}) \not{b} (\not{q} - y \not{p}) \gamma_\sigma (\not{q} + \not{p}(1-y)) \not{b} (\not{q} + \not{p}(1-y)) \right\}
\end{aligned} \tag{75}$$

We have 6 $q_\rho q_\sigma$ - integrals and one with no q's. We use the notations

$$\begin{aligned}
A1 &= \text{Tr} \left\{ \gamma_\rho \gamma_\mu \not{b} \gamma^\mu \gamma_\sigma \not{p} \not{b} \not{p} \right\} \\
&= (2-D) \left[-2p^2 b_\rho b_\sigma + 2pb(b_\rho p_\sigma + p_\rho b_\sigma) + g_{\rho\sigma}(b^2 p^2 - 2(pb)^2) \right] \\
A2 &= \text{Tr} \left\{ \gamma_\rho \gamma_\mu \not{b} \not{p} \gamma_\sigma \gamma^\mu \not{b} \not{p} \right\}
\end{aligned}$$

$$\begin{aligned}
&= -2b^2p^2g_{\rho\sigma} \\
A3 &= Tr \left\{ \gamma_\rho \gamma_\mu \not{b} \not{p} \gamma_\sigma \not{p} \not{b} \gamma^\mu \right\} \\
&= (2-D) \left(4pb b_\rho p_\sigma - 2p^2 b_\rho b_\sigma - 2b^2 p_\rho p_\sigma + b^2 p^2 g_{\rho\sigma} \right) \\
A4 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \gamma_\mu \gamma_\sigma \gamma^\mu \not{b} \not{p} \right\} \\
&= (2-D) \left(4pb p_\rho b_\sigma - 2p^2 b_\rho b_\sigma - 2b^2 p_\rho p_\sigma + b^2 p^2 g_{\rho\sigma} \right) \\
A5 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \gamma_\mu \gamma_\sigma \not{p} \not{b} \gamma^\mu \right\} \\
&= -2p^2 b^2 g_{\rho\sigma} \\
A6 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \not{p} \gamma_\sigma \gamma_\mu \not{b} \gamma^\mu \right\} \\
&= (2-D) [2pb b_\rho p_\sigma - 2p^2 b_\rho b_\sigma - 2(pb)^2 g_{\rho\sigma} + 2pb p_\rho b_\sigma + p^2 b^2 g_{\rho\sigma}] \\
A7 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \not{p} \gamma_\sigma \not{p} \not{b} \not{p} \right\} \\
&= 8(pb)^2 p_\rho p_\sigma + 2p^4 b_\rho b_\sigma - 4p^2 pb (b_\rho p_\sigma + p_\rho b_\sigma) - b^2 p^4 g_{\rho\sigma} \tag{76}
\end{aligned}$$

Integration over q yields (see eq. (113))

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(1)} (p) &= -\frac{i}{(4\pi)^2} \frac{1}{3Dp^2} 6 \int_0^1 dx \int_0^x dy y \frac{1}{[y(1-y)]^{3-\frac{D}{2}}} \\
&\left((1-y)^2 A1 - y(1-y) [A2 + A3 + A4] - y(1-y) A5 + y^2 A6 \right) \\
&- 6 \int_0^1 dx \int_0^x dy y \frac{i}{(4\pi)^2} \frac{1}{6p^4 [y(1-y)]^{2-\frac{D}{2}}} y^2 (1-y)^2 A7. \tag{77}
\end{aligned}$$

The integration over the Feynman parameters yields (Section B)

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(1)} (p) &= -\frac{i}{(4\pi)^2} \frac{1}{3Dp^2} \\
&\left((2A1 - [A2 + A3 + A4 + A5] + 2A6) - \frac{i}{(4\pi)^2} \frac{1}{6p^4} A7 \right). \tag{78}
\end{aligned}$$

By inserting the expressions (76) into (78) one gets

$$\prod_{\rho\sigma}^{\text{CONV}(1)} (p) = -\frac{i}{(4\pi)^2} \frac{1}{12p^2} \left\{ 2(2-D) \right.$$

$$\begin{aligned}
& \left[-2p^2 b_\rho b_\sigma + 2pb(b_\rho p_\sigma + p_\rho b_\sigma) + g_{\rho\sigma}(b^2 p^2 - 2(pb)^2) \right] \\
& + 2b^2 p^2 g_{\rho\sigma} \\
& - (2-D) \left(4pb b_\rho p_\sigma - 2p^2 b_\rho b_\sigma - 2b^2 p_\rho p_\sigma + b^2 p^2 g_{\rho\sigma} \right) \\
& - (2-D) \left(4pb p_\rho b_\sigma - 2p^2 b_\rho b_\sigma - 2b^2 p_\rho p_\sigma + b^2 p^2 g_{\rho\sigma} \right) \\
& + 2p^2 b^2 \\
& + 2(2-D) \left(2pb b_\rho p_\sigma - 2p^2 b_\rho b_\sigma - 2(pb)^2 g_{\rho\sigma} + 2pb p_\rho b_\sigma + p^2 b^2 g_{\rho\sigma} \right) \Big\} \\
& - \frac{i}{(4\pi)^2} \frac{1}{6p^4} \left(8(pb)^2 p_\rho p_\sigma + 2p^4 b_\rho b_\sigma \right. \\
& \left. - 4p^2 pb(b_\rho p_\sigma + p_\rho b_\sigma) - b^2 p^4 g_{\rho\sigma} \right) \\
& = -\frac{i}{(4\pi)^2} \frac{1}{6p^2} \left\{ (-4b^2 + \frac{8}{p^2}(pb)^2) p_\rho p_\sigma + 6p^2 b_\rho b_\sigma \right. \\
& \left. - 8(pb)(b_\rho p_\sigma + p_\rho b_\sigma) + (-b^2 p^2 + 8(pb)^2) g_{\rho\sigma} \right\} \tag{79}
\end{aligned}$$

8.6 Convergent Parts: Graph 2 and 3

Now we consider the graphs where the b_μ sources are on the same $\gamma_\rho - \gamma_\sigma$ fermion line. I.e. we use the amplitude in (56)

$$\begin{aligned}
\prod_{\rho\sigma} (p)^{(2)} &= - \int \frac{d^D q}{(2\pi)^D} \text{Tr} \left\{ \frac{1}{\not{q} + i\varepsilon} \gamma_\rho \frac{1}{[(\not{q} + \not{p}) + i\varepsilon]} \not{b} \gamma_\chi \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \not{b} \gamma_\chi \right. \\
&\quad \left. \frac{1}{[\not{q} + \not{p} + i\varepsilon]} \gamma_\sigma \right\}. \tag{80}
\end{aligned}$$

With the Feynman parameterization (99) one gets

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(2)} (p) &= -6 \int_0^1 dx \frac{x^2}{2} \int \frac{d^D q}{(2\pi)^D} \left[q^2 + p^2 x(1-x) \right]^{-4} \\
&\quad \text{Tr} \left\{ [\not{q} - x \not{p}] \gamma_\rho [\not{q} + \not{p}(1-x)] \not{b} [\not{q} + \not{p}(1-x)] \not{b} \right. \\
&\quad \left. [\not{q} + \not{p}(1-x)] \gamma_\sigma \right\} \\
&= -6 \int_0^1 dx \frac{x^2}{2} \left(\frac{i}{(4\pi)^2} \right) \left\{ \frac{1}{3p^2} \frac{1}{[x(1-x)]^{3-\frac{D}{2}}} \right. \\
&\quad \left. \frac{1}{D} \left((1-x)^2 (B1 + B2 + B3) \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& -x(1-x)(B4 + B5 + B6)) \\
& + \frac{1}{6p^4} \frac{1}{[x(1-x)]^{2-\frac{D}{2}}} \left(-x(1-x)^3 B7 \right) \Big\} \quad (81)
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= Tr \left\{ \gamma_\rho \gamma_\mu \not{b} \not{p} \not{b} \not{p} \gamma_\sigma \gamma^\mu \right\} \\
&= (2-D) \left[2pb(b_\rho p_\sigma - p_\rho b_\sigma) + g_{\rho\sigma}(2(pb)^2 - p^2 b^2) \right] \\
B_2 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \gamma_\mu \not{b} \not{p} \gamma_\sigma \gamma^\mu \right\} \\
&= -4b^2 p_\rho p_\sigma - 4p^2 b_\rho b_\sigma + 4pb(p_\rho b_\sigma + b_\rho p_\sigma) \\
&\quad + g_{\rho\sigma}(-4(pb)^2 + p^2 b^2(6-D)) \\
B_3 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \not{p} \not{b} \gamma_\mu \gamma_\sigma \gamma^\mu \right\} \\
&= (2-D) \left[2pb(p_\rho b_\sigma - b_\rho p_\sigma) + g_{\rho\sigma}(2(pb)^2 - p^2 b^2) \right] \\
B_4 &= Tr \left\{ \gamma_\rho \gamma_\mu \not{b} \gamma^\mu \not{b} \not{p} \gamma_\sigma \not{p} \right\} \\
&= (2-D) \left[2b^2 p_\rho p_\sigma - b^2 p^2 g_{\rho\sigma} \right] \\
B_5 &= Tr \left\{ \gamma_\rho \gamma_\mu \not{b} \not{p} \not{b} \gamma^\mu \gamma_\sigma \not{p} \right\} \\
&= (2-D) \left[-2b^2 p_\rho p_\sigma + 2pb(p_\rho b_\sigma + b_\rho p_\sigma) \right. \\
&\quad \left. + g_{\rho\sigma}(-2(pb)^2 + p^2 b^2) \right] \\
B_6 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \gamma_\mu \not{b} \gamma^\mu \gamma_\sigma \not{p} \right\} \\
&= (2-D) \left[2b^2 p_\rho p_\sigma - p^2 b^2 g_{\rho\sigma} \right] \\
B_7 &= Tr \left\{ \gamma_\rho \not{p} \not{b} \not{p} \not{b} \not{p} \gamma_\sigma \not{p} \right\} \\
&= (8(pb)^2 - 2p^2 b^2) p_\rho p_\sigma - 2(pb)p^2(b_\rho p_\sigma + p_\rho b_\sigma) \\
&\quad + g_{\rho\sigma}[p^4 b^2 - 2p^2(pb)^2] \quad (82)
\end{aligned}$$

In eqs. (81) and (82) γ_χ have been dropped because we consider only convergent integrals.

From eqs. (113), (81) and (114) we get

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(2)} (p) &= -\frac{i}{(4\pi)^2} \frac{1}{3Dp^2} \left\{ \frac{1}{2} B1 + \frac{1}{2} B2 + \frac{1}{2} B3 \right. \\
&\quad \left. - B4 - B5 - B6 \right\} + \frac{i}{(4\pi)^2} \frac{1}{12p^4} B7 \quad (83)
\end{aligned}$$

Now we evaluate eq. (83) by using the 7 traces (82)

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(2)} (p) = & -\frac{i}{(4\pi)^2} \frac{1}{3Dp^2} \left[\right. \\
& \frac{1}{2}(2-D) \left(2pb(b_\rho p_\sigma - p_\rho b_\sigma) + g_{\rho\sigma}[2(pb)^2 - p^2 b^2] \right) \\
& + \frac{1}{2} \left(-4b^2 p_\rho p_\sigma - 4p^2 b_\rho b_\sigma + 4pb(p_\rho b_\sigma + b_\rho p_\sigma) \right. \\
& \left. + g_{\rho\sigma}[-4(pb)^2 + p^2 b^2(6-D)] \right) \\
& + \frac{1}{2}(2-D) \left(2pb(p_\rho b_\sigma - b_\rho p_\sigma) + g_{\rho\sigma}[2(pb)^2 - p^2 b^2] \right) \\
& - (2-D) \left(2b^2 p_\rho p_\sigma - b^2 p^2 g_{\rho\sigma} \right) \\
& - (2-D) \left(-2b^2 p_\rho p_\sigma + 2pb(p_\rho b_\sigma + b_\rho p_\sigma) \right. \\
& \left. + g_{\rho\sigma}[-2(pb)^2 + p^2 b^2] \right) \\
& \left. - (2-D) \left(2b^2 p_\rho p_\sigma - p^2 b^2 g_{\rho\sigma} \right) \right] \\
& + \frac{i}{(4\pi)^2} \frac{1}{12p^4} \left[(8(pb)^2 - 2p^2 b^2) p_\rho p_\sigma - 2(pb)p^2(b_\rho p_\sigma + p_\rho b_\sigma) \right. \\
& \left. + g_{\rho\sigma}[p^4 b^2 - 2p^2(pb)^2] \right]. \tag{84}
\end{aligned}$$

Finally

$$\begin{aligned}
\prod_{\rho\sigma}^{\text{CONV}(2)} (p) = & -\frac{i}{(4\pi)^2} \frac{1}{12p^2} \text{Tr} \left\{ 4b^2 p_\rho p_\sigma - \frac{8}{p^2} (pb)^2 p_\rho p_\sigma - 2p^2 b_\rho b_\sigma \right. \\
& \left. + 8pb(p_\rho b_\sigma + b_\rho p_\sigma) - 8(pb)^2 g_{\rho\sigma} \right\}. \tag{85}
\end{aligned}$$

By adding the contribution of graph n.3 we get

$$\prod_{\rho\sigma}^{\text{CONV}(2)} (p) + \prod_{\rho\sigma}^{\text{CONV}(3)} (p)$$

$$\begin{aligned}
&= -\frac{i}{(4\pi)^2} \frac{1}{6p^2} \text{Tr} \left\{ 4b^2 p_\rho p_\sigma - \frac{8}{p^2} (pb)^2 p_\rho p_\sigma - 2p^2 b_\rho b_\sigma \right. \\
&\quad \left. + 8pb(p_\rho b_\sigma + b_\rho p_\sigma) - 8(pb)^2 g_{\rho\sigma} \right\}. \tag{86}
\end{aligned}$$

From eq. (79) we get

$$\sum_{j=1,2,3} \prod_{\rho\sigma}^{\text{conv}(j)} (p) = -\frac{i}{(4\pi)^2} \frac{1}{6} [4b_\rho b_\sigma - b^2 g_{\rho\sigma}]. \tag{87}$$

Remember the POLE parts of graphs (1), (2) and (3) from eq. (69) and the LOG part in eq. (73)

$$\prod_{\rho\sigma}^{\text{POLE}} (p) = -\frac{i}{(4\pi)^2} \frac{1}{6} (-2b_\rho b_\sigma - b^2 g_{\rho\sigma}) \tag{88}$$

$$\sum_{j=1}^3 \prod_{\rho\sigma}^{\text{LOG}(j)} (p) = -\frac{i}{(4\pi)^2} \frac{1}{6} (-2b_\rho b_\sigma + 2b^2 g_{\rho\sigma}). \tag{89}$$

The final result based on eqs. (87), (88) and (89) is then

$$\prod_{\rho\sigma} (p) = 0. \tag{90}$$

The result in eq. (90) indicates that the vacuum described by the b_μ source in the action (2) has no effect on massless QED, beside the birefringence given by the Chern-Simons (1) of the ABJ anomaly.

We make explicit the above argument. By considering the chiral transformation

$$\psi \rightarrow e^{i\alpha\gamma_5} \psi \tag{91}$$

we get a Ward identity on the vertex functional generated from the path integral functional of eq. (13)

$$\partial_\mu \frac{\delta\Gamma}{\delta b_\mu(x)} - i \frac{\delta\Gamma}{\delta\psi} \gamma_5 \psi + i \bar{\psi} \gamma_5 \frac{\delta\Gamma}{\delta\bar{\psi}} = \Delta(x) \cdot \Gamma \tag{92}$$

where $\Delta \cdot \Gamma$ is the insertion of the ABJ anomaly

$$\Delta(x) = \frac{e^2}{(4\pi)^2} \varepsilon_{\mu\nu\rho\sigma} \partial^\mu A^\nu \partial^\rho A^\sigma. \tag{93}$$

From a perturbative point of view, the insertion Δ brings a e^2 factor in the e series expansion.

Now we can differentiate eq. (92) with respect to $b_\nu(y), A_\rho(v), A_\sigma(w)$ and fix the external momenta $(-p-k, 0, p, k)$. With short hand notation eq. (92) becomes

$$(p+k)^\mu \Gamma_{\mu\nu\rho\sigma}(-p-k, 0, p, k) = \mathcal{O}(e^6) \quad (94)$$

i.e.

$$(p+k)^\mu \Gamma_{\mu\nu\rho\sigma}(-p-k, 0, p, k) \Big|_{\text{one-loop}} = 0, \quad (95)$$

since the LHE of eq. (95) is $\mathcal{O}(e^2)$ and therefore a finite value could not match the RHS.

We now differentiate the above equation with respect to p_τ and then put $p = -k$. Thus finally we get at one loop

$$\Gamma_{\mu\nu\rho\sigma}^{(1)}(-p-k, 0, p, k) \Big|_{p=-k} = 0, \quad (96)$$

i.e. the result of our explicit calculation in eq. (90).

Our result (90) disagrees with the conclusions of a similar investigation in Refs. [5] and [6].

9 Conclusions

By using a new approach to γ_5 problem in dimensional regularization we computed the Chern-Simons effective action term in presence of a local CPT- and Lorentz- violating term in the Lagrangian. No ambiguities are present in the calculation. The calculations are straightforward application of the algebra developed in Refs. [7] and [8].

We have analyzed the different limits $m^2 = 0$ and $p^2 = 0$. For $m^2 = 0$ the CPT-violating action-term can be removed by a simple chiral transformation. The anomaly in the axial current is the only relic of the perturbative expansion in powers of b_μ . The result is in agreement with the approach based on Pauli-Villars regularization. However we understand that other regularization procedures might yield different results [4].

We obtain that the Chern-Simons term is zero at one loop level for $m^2 \neq 0, p^2 = 0$ in the dimensional regularization scheme. Thus we confirm the result of Pauli-Villars regularization.

For virtual photons ($p^2 \ll m^2$) the two-point function gets a finite contribution by the presence of the CPT- and Lorentz- violating term. The coefficient of the p^2 is in agreement with previous gauge invariant calculations (via Pauli-Villars). Some authors find a non-zero limit for $p^2 = 0$, i.e. a finite Chern-Simons term (1). This result is obtained by abandoning dimensional regularization as in Refs. [3], [4],[9].

We have devoted a consistent part of the paper to the explicit calculation of the photon self-energy at second order in the expansion in b_μ for the case $m^2 = 0$. The interest lies in the question whether, by the removal of the CPT- and Lorentz- violating term, some other relics remain beside the Chern-Simons term. The places where to look for, are the superficially divergent amplitudes as the box-diagram in the photon self-energy. After an excruciating calculation we found zero: a result expected if ABJ anomaly is the only terms present in the Ward identity (92) associated to local chiral transformations.

Acknowledgements

One of us (RF) gratefully thanks Roman Jackiw for suggesting him the present subject and for challenging discussions. The warm and stimulating hospitality of CTP-MIT is gratefully acknowledged. Part of this work has been done at the Department of Physics of the University of Pisa and of the INFN, Sezione di Pisa, to which he is very grateful.

A Collection of Standard Integrals

In order to improve the presentation of the paper we remind some Feynman formula

$$\frac{1}{A_1 \dots A_n} = (n-1)! \int_0^1 du_1 \int_0^{u_1} du_2 \dots \int_0^{u_{n-2}} du_{n-1} \left[A_1 + u_1(A_2 - A_1) + \dots + u_{n-1}(A_n - A_{n-1}) \right]^{-n} \quad (97)$$

From eqs. (54) and (56) we have the denominators

$$\begin{aligned} & \frac{1}{q^2 q^2 (q+p)^2 (q+p)^2} \\ &= 6 \int_0^1 dx \int_0^x dy \, y \left[(q+yp)^2 + p^2 y(1-y) \right]^{-4} \end{aligned} \quad (98)$$

and

$$\begin{aligned} & \frac{1}{q^2(q+p)^2(q+p)^2(q+p)^2} \\ &= 6 \int_0^1 dx \frac{x^2}{2} \left[(q+xp)^2 + p^2x(1-x) \right]^{-4}. \end{aligned} \quad (99)$$

Use Dimensional Integrals. In Minkowski space we have

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2)^k} = i(-)^k \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(k - \frac{D}{2})}{\Gamma(k)} \frac{1}{(m^2)^{(k - \frac{D}{2})}} \quad (100)$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - m^2)^k} = -i(-)^k \frac{D}{2} \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(k - 1 - \frac{D}{2})}{\Gamma(k)} \frac{1}{(m^2)^{(k - 1 - \frac{D}{2})}} \quad (101)$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{(q^2)^2}{(q^2 - m^2)^k} = i(-)^k \frac{D(D+2)}{4} \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(k - 2 - \frac{D}{2})}{\Gamma(k)} \frac{1}{(m^2)^{(k - 2 - \frac{D}{2})}} \quad (102)$$

$$\gamma = 0.5772156649 \quad (103)$$

For $D \rightarrow 4$ we get $(\Gamma(z) \sim z^{-1} - \gamma + \mathcal{O}(z))$

$$\begin{aligned} & \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - m^2)^3} = \frac{i}{(4\pi)^2} \left[-\frac{2}{D-4} - \gamma - \frac{1}{2} - \ln\left(\frac{m^2}{4\pi}\right) \right] \\ &= -\frac{i}{(4\pi)^2} \left[\frac{2}{D-4} + \gamma + \frac{1}{2} + \ln\left(\frac{m^2}{4\pi}\right) \right] + \mathcal{O}(D-4) \end{aligned} \quad (104)$$

and

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2)^3} = -\frac{i}{(4\pi)^2} \frac{1}{2m^2} \quad (105)$$

Derivative with respect to m^2 For $D \rightarrow 4$ we get

$$\int \frac{d^D q}{(2\pi)^D} \frac{q^2}{(q^2 - m^2)^4} = -\frac{i}{(4\pi)^2} \frac{1}{3m^2} \quad (106)$$

and

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2)^4} = \frac{i}{(4\pi)^2} \frac{1}{6m^4} \quad (107)$$

From eq. (104)

$$\begin{aligned} & \int \frac{d^D q}{(2\pi)^D} \frac{q^4}{(q^2 - m^2)^4} \\ &= -\frac{i}{(4\pi)^2} \left[\frac{2}{D-4} + \gamma + \frac{5}{6} + \ln\left(\frac{m^2}{4\pi}\right) \right] + \mathcal{O}(D-4) \end{aligned} \quad (108)$$

B Integrals over Feynman parameters

For the log parts we need the integrals in (98) and (99)

$$\begin{aligned} 6 \int_0^1 dx \int_0^x dy y \ln(y) &= -\frac{5}{6} \\ 6 \int_0^1 dx \int_0^x dy y \ln(1-y) &= -\frac{5}{6} \end{aligned} \quad (109)$$

Then

$$6 \int_0^1 dx \int_0^x dy y \ln(y(1-y)) = -\frac{5}{3}. \quad (110)$$

Similarly

$$\begin{aligned} 6 \int_0^1 dx \frac{x^2}{2} \ln(x) &= -\frac{1}{3} \\ 6 \int_0^1 dx \frac{x^2}{2} \ln(1-x) &= -\frac{11}{6} \end{aligned} \quad (111)$$

i.e.

$$6 \int_0^1 dx \frac{x^2}{2} \ln(x(1-x)) = -\frac{13}{6} \quad (112)$$

B.1 Integrals over Feynman parameters for $\prod^{CONV(1)}$

We need some elementary integrals

$$\begin{aligned} \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{[q^2 + p^2 y(1-y)]^4} &= \frac{i}{(4\pi)^2} \frac{1}{3p^2 y(1-y)} \\ \int \frac{d^D q}{(2\pi)^D} [q^2 + p^2 y(1-y)]^{-4} &= \frac{i}{(4\pi)^2} \frac{1}{6p^4 [y(1-y)]^2} \\ 6 \int_0^1 dx \int_0^x dy &= 3 \\ 6 \int_0^1 dx \int_0^x dy y &= 1 \\ 6 \int_0^1 dx \int_0^x dy \frac{y^2}{1-y} &= 2 \end{aligned} \quad (113)$$

B.2 Integrals over Feynman parameters: $\prod^{CONV(2)}$

Integrals:

$$6 \int_0^1 dx \frac{x^2}{2} = 1$$

$$6 \int_0^1 dx \frac{x^2}{2} \frac{1-x}{x} = \frac{1}{2} \quad (114)$$

C Chern-Simons at generic p^2, m^2

This Appendix includes the computation of the photon polarization tensor at first order in the parameter b_μ for every values of the momentum in the massive case. We follow the algebra of Section 5. We have two graphs and we consider them one by one.

$$\Gamma_{\rho\sigma} = -i \int \frac{d^D q}{(2\pi)^D} \frac{\text{Tr}(\not{q} + m) \gamma_\alpha \gamma_\chi (\not{q} + m) \gamma_\rho (\not{q} + \not{p} + m) \gamma_\sigma}{(q^2 - m^2)^2 ((q + p)^2 + m^2)}. \quad (115)$$

By introducing the Feynman parametrization and performing the usual translation on the integration variables we easily arrive at

$$\begin{aligned} \Gamma_{\mu\rho\sigma}^{\text{DIV}}(k, p) &= -i \frac{1}{2} \int_0^1 dx x \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} \\ &\quad \left((1-x)p^\iota g^{\alpha\beta} + (-)xp^\beta g^{\alpha\iota} + (1-x)p^\alpha g^{\beta\iota} \right) \\ &\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(\Delta) \right] \\ &= -i \frac{1}{2} \int_0^1 dx x \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \left(2(1-x)(2-D) + x(6-D) \right) \\ &\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi + \ln(\Delta) \right], \end{aligned} \quad (116)$$

where the algebra of the gamma's and γ_χ is performed according to the rules of Section 2. We separately integrate over x the x -independent part and $\ln \Delta(x)$

$$\begin{aligned} \Gamma_{\mu\rho\sigma}^{\text{DIV}}(k, p) &= -i \frac{1}{2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \frac{2}{3} (4-D) \\ &\quad \left(-\frac{i}{(4\pi)^2} \right) \left[\frac{2}{D-4} + \gamma + 2 - \ln 4\pi \right] \\ &\quad -i \frac{1}{2} \int_0^1 dx \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} (-4x + 6x^2) \\ &\quad \left(-\frac{i}{(4\pi)^2} \right) \ln(m^2 - p^2 x(1-x)) \end{aligned} \quad (117)$$

We use the integrals by parts

$$\begin{aligned} \int_0^1 dx x \ln(m^2 - p^2 x(1-x)) &= \frac{1}{2} \int_0^1 dx x^2 \frac{p^2(1-2x)}{m^2 - p^2 x(1-x)} \\ \int_0^1 dx x^2 \ln(m^2 - p^2 x(1-x)) &= \frac{1}{3} \int_0^1 dx x^3 \frac{p^2(1-2x)}{m^2 - p^2 x(1-x)} \end{aligned} \quad (118)$$

Thus we get finally

$$\begin{aligned}\Gamma_{\mu\rho\sigma}^{\text{DIV}}(k, p) = & -i\frac{1}{2}\left(-\frac{i}{(4\pi)^2}\right)Tr \left\{\gamma_\mu\gamma_\chi\gamma_\rho\gamma_\sigma \not{p}\right\}\left(-\frac{4}{3}\right. \\ & \left. + \int_0^1 dx x^2 2(-1+x) \frac{p^2(1-2x)}{m^2 - p^2 x(1-x)}\right)\end{aligned}\quad (119)$$

The convergent integral can be evaluated as in eq.(38) with $D = 4$.

$$\begin{aligned}\Gamma_{\mu\rho\sigma}^{\text{CONV}}(p) = & -i2\frac{1}{(4\pi)^2}\int_0^1 dx x \frac{1}{m^2 - p^2(x-x^2)} \\ & Tr \left\{\gamma_\mu\gamma_\chi [(1-x) \not{p} + m] \gamma_\rho [-x \not{p} + m] \gamma_\sigma \right. \\ & \left. [(1-x) \not{p} + m]\right\} \\ = & -\frac{1}{(4\pi)^2}\int_0^1 dx x \frac{1}{m^2 - p^2(x-x^2)} \\ & \left(T r \left\{\gamma_\mu\gamma_\chi (1-x) \not{p} \gamma_\rho (-x) \not{p} \gamma_\sigma (1-x) \not{p}\right\}\right. \\ & \left. + m^2[2(1-x) + x] Tr \left\{\gamma_\mu\gamma_\chi\gamma_\rho\gamma_\sigma \not{p}\right\}\right) \\ = & -\frac{1}{(4\pi)^2}\int_0^1 dx x \frac{p^2 x(1-x)^2 + m^2(2-x)}{m^2 - p^2(x-x^2)} \\ & Tr \left\{\gamma_\mu\gamma_\chi\gamma_\rho\gamma_\sigma \not{p}\right\}.\end{aligned}\quad (120)$$

Now we add the expressions in (119) and (120) and introduce a factor 2 for the crossed graph.

$$\begin{aligned}\Gamma_{\mu\rho\sigma}(p) = & -\frac{2}{(4\pi)^2}Tr \left\{\gamma_\mu\gamma_\chi\gamma_\rho\gamma_\sigma \not{p}\right\}\left(-\frac{2}{3}\right. \\ & \left. + \int_0^1 dx \frac{1}{m^2 - p^2(x-x^2)}\right. \\ & \left.\left[p^2 x^2(1-x)^2 + m^2 x(2-x) + p^2 x^2(-1+x)(1-2x)\right]\right) \\ = & -\frac{2}{(4\pi)^2}Tr \left\{\gamma_\mu\gamma_\chi\gamma_\rho\gamma_\sigma \not{p}\right\}\left(-\frac{2}{3} + \int_0^1 dx \frac{1}{m^2 - p^2(x-x^2)}\right. \\ & \left.\left[m^2(2x-x^2) + p^2 x^2(-x^2+x)\right]\right)\end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \left(-1 \right. \\
&\quad \left. + m^2 \int_0^1 dx \frac{2x}{m^2 - p^2(x - x^2)} \right). \tag{121}
\end{aligned}$$

We use the identity

$$\int_0^1 dx \frac{-1 + 2x}{m^2 - p^2(x - x^2)} = \int_0^1 dx \frac{1}{p^2} \frac{d}{dx} \ln(m^2 - p^2(x - x^2)) = 0. \tag{122}$$

Thus eq. (121) can be written

$$\Gamma_{\mu\rho\sigma}(p) = -\frac{2}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \left(-1 + m^2 \int_0^1 dx \frac{1}{m^2 - p^2(x - x^2)} \right). \tag{123}$$

For $p^2 \ll m^2$ we have

$$\Gamma_{\mu\rho\sigma}(p) = -\frac{2}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \frac{1}{6} \frac{p^2}{m^2} \tag{124}$$

which is in agreement with eq. (52).

For $p^2 \gg m^2$ we must restore Feynman path in

$$\Gamma_{\mu\rho\sigma}(p) = -\frac{2}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \left(-1 + m^2 \int_0^1 dx \frac{1}{m^2 - i\epsilon - p^2(x - x^2)} \right). \tag{125}$$

In the limit $m^2 \rightarrow 0$ the factor in front kills the logarithmic behavior of the integral. Thus finally

$$\lim_{m^2=0} \Gamma_{\mu\rho\sigma}(p) = \frac{2}{(4\pi)^2} \text{Tr} \left\{ \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \not{p} \right\} \tag{126}$$

as in eq. (39).

References

- [1] D. Colladay and V. A. Kostelecky, Phys. Rev. D **55**, 6760 (1997) [hep-ph/9703464].
- [2]
- [2] D. Colladay and V. A. Kostelecky, Phys. Rev. D **58**, 116002 (1998) [hep-ph/9809521].

- [3] R. Jackiw and V. A. Kostelecky, “Radiatively induced Lorentz and CPT violation in electrodynamics,” *Phys. Rev. Lett.* **82**, 3572 (1999) [hep-ph/9901358].
- [4] R. Jackiw, “When radiative corrections are finite but undetermined,” *Int. J. Mod. Phys. B* **14**, 2011 (2000) [hep-th/9903044].
- [5] B. Altschul, *Phys. Rev. D* **69**, 125009 (2004) [hep-th/0311200].
- [6] B. Altschul, “Gauge invariance and the Pauli-Villars regulator in Lorentz- and CPT-violating electrodynamics,” *Phys. Rev. D* **70**, 101701 (2004) [hep-th/0407172].
- [7] R. Ferrari, “Managing γ_5 in Dimensional Regularization and ABJ Anomaly,” arXiv:1403.4212 [hep-th].
- [8] R. Ferrari, “Managing γ_5 in Dimensional Regularization II: the Trace with more γ_5 ,” arXiv:1503.07410 [hep-th].
- [9] J. M. Chung and P. Oh, “Lorentz and CPT violating Chern-Simons term in the derivative expansion of QED,” *Phys. Rev. D* **60**, 067702 (1999) [hep-th/9812132].
- [10] J. M. Chung, “Radiatively induced Lorentz and CPT violating Chern-Simons term in QED,” *Phys. Lett. B* **461**, 138 (1999) [hep-th/9905095].
- [11] M. Perez-Victoria, “Exact calculation of the radiatively induced Lorentz and CPT violation in QED,” *Phys. Rev. Lett.* **83**, 2518 (1999) [hep-th/9905061].
- [12] M. Perez-Victoria, “Physical (ir)relevance of ambiguities to Lorentz and CPT violation in QED,” *JHEP* **0104**, 032 (2001) [hep-th/0102021].